GROUP CLASSIFICATION OF SOLUTIONS OF THE HOPF EQUATION

V. L. Katkov

Zhurnal Prikladnoi Mekhaniki i Tekhnicheskoi Fiziki, No. 6, pp. 105-106, 1965.

A group classification of the solutions of the Hopf equation in terms of the form of the "viscosity" coefficient is given, together with the various solutions of this equation.

The problem consists in finding the principal group of admissible transforms for the equation

$$u_t + uu_x = (\varepsilon u_x)_x \qquad (1)$$

and expanding this group on the basis of a choice of the particular forms of the function $\varepsilon(u)$, where examples are given of the group classification of the equations of nonlinear heat conduction, adiabatic motions of a gas, etc.

Equation 1 admits the transformations of translation $u = u + a$, $x = at + x$, $t = t$ and expansion $u = u/b$, $x = bx$, $t = b^2t$, so that any linear transformation $u_1 = pu + q$ is admissible with respect to u. Referred to the function $\varepsilon(u)$, this transformation will have the form $\varepsilon_1(u) = \varepsilon(pu + q)$. Two equations of the form of (1) will be termed equivalent if one of them turns into the other after Galilean transformation of translation and expansion.

The calculation of the principal group admitted by a system of equations is described in [1]; here, we give the final result in the form of base operators of the corresponding Lie algebra.

1. Let $\varepsilon(u)$ be an arbitrary function. Equation (1) admits only two operators

$$X_1 = \frac{\partial}{\partial t}, \qquad X_2 = \frac{\partial}{\partial x} \qquad (2)$$

2. If, however, $\varepsilon(u) = e^u$, then equation (1) admits a wider group; to the operators (2) is then added

$$X_3{}^1 = t\frac{\partial}{\partial t} + (x+t)\frac{\partial}{\partial x} + \frac{\partial}{\partial u}$$

3. For $\varepsilon(u) = u^{2m}$, the principal group will also be generated by three operators, but instead of X^1 we have

$$X_3{}^2 = 2(m-1)t\frac{\partial}{\partial t} + (2m-1)x\frac{\partial}{\partial x} + u\frac{\partial}{\partial u}$$

4. $\varepsilon(u) = 1$. In this case, to the operators (2) there are immediately added three new ones.

$$X_3{}^3 = t\frac{\partial}{\partial x} + \frac{\partial}{\partial u}, \quad X_4 = 2t\frac{\partial}{\partial t} + x\frac{\partial}{\partial x} - u\frac{\partial}{\partial u},$$

$$X_5 = t^2\frac{\partial}{\partial t} + tx\frac{\partial}{\partial x} + (x-ut)\frac{\partial}{\partial u}$$

5. If $\varepsilon = 0$, the Lie algebra becomes infinite-dimensional: equation (1) admits any operator of the form

$$X = \psi(t, x, u)\frac{\partial}{\partial t} + [u\psi + tf(u, x-ut) + \varphi(u, x-ut)]\frac{\partial}{\partial x} + f\frac{\partial}{\partial u}$$

with arbitrary f, φ, ψ functions.

The construction of invariant solutions on these groups requires a knowledge of the optimal systems of each group, i.e., should be limited to essentially distinct solutions [1]. Neglecting the computations, we present the results, i.e., the optimal systems of each group and the form of the solution (see table which shows the operators of the subgroups and the form of the solution).

$\varepsilon = \varepsilon(u)$	X_1 X_2	$u = U(x)$ $u = U(t)$
$\varepsilon = e^u$	X_1, X_2 X_3^1	$u = \ln t + U(\lambda), \quad \lambda = x/t - \ln t$
$\varepsilon = u^{2m}$	$m = 1/2$ X_2 $X_1 + X_2$ $X_2 + X_3{}^2$ $m = 1$ $X_1, \quad X_1 + X_2$ $X_1 + X_3{}^2$ for any m $X_2, \quad X_1 + X_2$ $X_3{}^2$	$u = U(x-t)$ $u = t^{-1}U(\lambda), \ \lambda = x + \ln t$ $u = e^t U(\lambda), \quad \lambda = xe^{-t}$ $u = t^{\frac{1}{2(m-1)}} U(\lambda), \quad \lambda = xt^{-\frac{2m-1}{2(m-1)}}$
$\varepsilon = 1$	X_1, X_2 X_4 $X_1 + X_3{}^3$ $X_4 + X_5$ $\alpha X_1 + \beta X_2 + X_5$	$u = t^{-1/2}U(\lambda), \ \lambda = xt^{-1/2}$ $u = t + U(\lambda), \ \lambda = x - 1/2\, t^2$ $u = x(t+2)^{-1} + 1/x\ U(\lambda), \ \lambda = x[t(t+2)]^{-1/2}$ $\alpha \neq 0$ (say $\alpha = 1$) $u = \beta + (t^2+1)^{-1/2}[\lambda t + U(\lambda)]$ $\lambda \neq (x - \beta t)(t^2+1)^{-1/2}$ $\alpha = 0, u = \beta t^{-2} + \lambda + t^{-1}U(\lambda), \lambda = x/t + \beta t^{-2}$

L. V. Ovsyannikov has noted that the solution of the group classification problem for an equation or the equivalent system of first-order equations may lead to different results. In our case, classification was performed in two ways: for equation (1), and for an equivalent system of two first-order equations. The results were found to be identical.

REFERENCES

1. L. V. Ovsyannikov, Group Properties of Differential Equations [in Russian], Novosibirsk, 1962.

25 June 1965 Leningrad